perturbation of the gyroscopic-force type.

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REFERENCES

1. HERTZ H., Die Prinziepien der Mechanik in neuem Zusammenhänge dargestellt: Ges. Werke. B. 3 Leipzig: Barth, 1894.
2. APPEL' P., Theoretical Mechanics, 2, Moscow, Fizmatgiz, 1960.
3. SYNGE J.L., Tensor Methods in Dynamics, /Russian translation/, Moscow, Gostekhizdat, 1947.
4. CARATHÉODORY C., Der Schlitten, Z. angew. Math. und Mech. 13, 2, 1933.
5. NEIMARK YU.I. and FUFAYEV N.A., Dynamics of Non-holonomic Systems. Moscow, Nauka, 1967.
6. KARAPETYAN A.V., On realizing non-holonomic constraints by forces of viscous friction and stability of Celtic stones. PMM, 45, 1, 1981.
7. KOZLOV V.V., Dynamics of systems with non-integrable constraints. III Vestn. MGU, Ser.1, Matematika, mekhanika, 3, 1983.
8. KOZLOV V.V., Realization of non-integrable constraints in classical mechanics.Dokl. A ad. Nauk SSSR, 272, 3, 1983.
9. ZHURAVLEV V.F. and KLIMOV D.M., The Solid State Wave Gyroscope. Moscow, Nauka, 1985.
10. ARNOL'D V.I., Additional Chapters in the Theory of Ordinary Differential Equations. Moscow, Nauka, 1978.
11. TATARINOV YA.V., Lectures on Classical Dynamics, Moscow, Izd-vo MGU, 1984.
12. MOZER YU., On expanding conditionally periodic motions in convergent power series. Uspekhi Mat. Nauk, 24, 2, 1969.

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ON THE CHANGE IN THE ADIABATIC INVARIANT ON CROSSING A SEPARATRIX IN SYSTEMS WITH TWO DEGREES OF FREEDOM*

A.I. NEISHTADT

Hamiltonian systems with two degrees of freedom are studied. One degree of freedom corresponds to rapid motion, and the other to slow motion. The phase point intersects the separatrix of the rapid motion. Formulas are obtained for the change in the adiabatic invariant during this crossing. An example is solved, dealing with the change in the adiabatic invariant of an asteroid near the 3:1 resonance with Jupiter.

1. Formulation of the problem. A number of problems of the theory of oscillations lead to Hamiltonian systems with a Hamiltonian of the form $H = H(p, q, y, x)$, where $q, \varepsilon^{-1}x$ are the coordinates, p, y the associated moments, $\varepsilon > 0$ is a small parameter and $H \in C^\infty$. The variables p, q will be called rapid, and y, x slow. The Hamiltonian system for p, q with $(y, x) = \text{const}$ will be called rapid or unperturbed. The Hamiltonian of the type shown characterizes, e.g. the motion of an asteroid in the bounded three-body problem near a resonance.

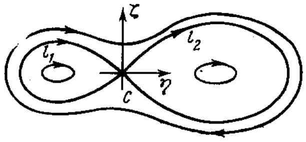

Fig.1

Below we assume that the phase plane of the rapid system contains the separatrices shown in Fig.1 for all values of the slow variables under cosideration. When the slow variables are varied, the phase point intersects the separatrix. The motion away from the sepatrix is characterized by a quantity which is preserved with a high degree of accuracy, namely the adiabatic invariant (AI) /1/. The neighbourhood of the sepratrices

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represents a zone of non-adiabaticity, and the AI changes when passing through this zone. Asymptotic formulas describing this change are obtained in this paper, and a formula describing the change in the AI of a resonant asteroid is derived from them as a corollary.

The change in the AI on crossing a separatrix was studied earlier in a Hamiltonian system with one degree of freedom, whose parameter varied smoothly with time. The Hamiltonian $H = H(p, q, \varepsilon t)$. The corresponding formula was obtained in /2, 3/, and for the special case of a pendulum in a smoothly varying gravitational field it was first obtained in /4/.

2. The adiabatic approximation. Let $I = I(H, y, x)$ be the "action" variable /1/ of the rapid system for any region not containing the separatrices, and let $\Phi(I, y, x)$ be the Hamiltonian H expressed in terms of I, y, x. In the case of motion away from the separatrices at times of the order of $1/\varepsilon$, the quantity I remains constant with an accuracy of the order of $O(\varepsilon)$, and the variation in y, x is described by a Hamiltonian system with Hamiltonian $\varepsilon\Phi(I, y, x)$, where y, x are conjugate variables, $I = \text{const}$ (the slow system) /5/. The approximation is called adiabatic, and the "action" is I - AI. We have the useful identities $\partial\Phi/\partial\alpha = \langle\partial H/\partial\alpha\rangle$, $\alpha = y, x$, where the angle brackets denote averaging over the phase of the unperturbed motion.

We shall call the following quantity the improved AI:

$$J = J(p, q, y, x) = I + \varepsilon u(p, q, y, x) \tag{2.1}$$

$$u = \frac{1}{4\pi}\left[\int_0^T\left(\frac{\partial H}{\partial y}\int_0^t \frac{\partial H}{\partial x}\,d\sigma\right)dt - \int_0^T\left(\frac{\partial H}{\partial x}\int_0^t \frac{\partial H}{\partial y}\,d\sigma\right)dt\right]$$

The integrals are taken along the phase trajectory of the rapid system passing through the point (p, q), t, σ is the time of motion measured from the instant of passage through this point, and T is the period. The quantity J represents the improved first approximation in the method of averaging /6/. In the case of motion away from the separatrices, at times of the order of $1/\varepsilon$, the quanity J will remain constant to within $O(\varepsilon^2)$.

The formula (2.1) can be obtained as follows. In the rapid system the action-angle variables $I, \varphi \bmod 2\pi$ are introduced by a symplectic change of variables, with the generating function $W(I, q, y, x)$. In the complete system we carry out the symplectic change of variables $p, q, y, \varepsilon^{-1}x \mapsto I, \bar\varphi, \bar y, \varepsilon^{-1}\bar x$ with the generating function $\varepsilon^{-1}\bar y x + W(I, q, \bar y, x)$. In the new variables the Hamiltonian takes the form

$$F = \Phi(I, \bar y, \bar x) + \varepsilon F_1(I, \bar\varphi, \bar y, \bar x) + O(\varepsilon^2) \tag{2.2}$$

$$F_1 = -\frac{\partial\Phi}{\partial x}\frac{\partial W}{\partial y} + \frac{\partial H}{\partial y}\frac{\partial W}{\partial x}$$

$$\frac{\partial W}{\partial\alpha} = -\frac{1}{\omega}\int_0^\varphi\left(\frac{\partial H}{\partial\alpha} - \left\langle\frac{\partial H}{\partial\alpha}\right\rangle\right)d\varphi, \quad \omega = \frac{\partial\Phi}{\partial I}, \quad \alpha = y, x$$

The angle φ is measured from any straight line $q = \text{const}$.

In order to obtain the improved AI, we carry out the symplectic change of variables $I, \bar\varphi, \bar y, \varepsilon^{-1}\bar x \mapsto J, \psi, Y, \varepsilon^{-1}X$, which is nearly identical and such that the Hamiltonian expressed in the new variables contains the phase ψ only in terms of order ε^2. The value of J is determined, to within to terms of order ε^2, by the formula

$$J = I + \varepsilon u(p, q, y, x), \quad u = \frac{F_1 - \langle F_1\rangle}{\omega} - \frac{\partial I}{\partial y}\frac{\partial W}{\partial x}$$

The function u is invariant with respect to the choice of the point on the unperturbed trajectory from which the angle φ is measured. We can therefore assume that φ is measured from the point (p, q). Then we have

$$u = -\frac{\langle F_1\rangle}{\omega} = \frac{1}{\omega}\left\langle\frac{\partial H}{\partial x}\right\rangle\left\langle\frac{\partial W}{\partial y}\right\rangle - \frac{1}{\omega}\left\langle\frac{\partial H}{\partial y}\frac{\partial W}{\partial x}\right\rangle$$

Substituting here $\partial W/\partial x$, $\partial W/\partial y$ from (2.2) and carrying out identical transformations, we obtain (2.1).

3. Passage across the separatrix in the adiabatic approximation. The phase pattern of the rapid system (Fig.1) shows that the separatrices l_1 and l_2 dividing the plane into the regions $G_i = G_i(y, x)$, $i = 1, 2, 3$, pass through the non-degenerate singular saddle point C. Let us denote by $h_C = h_C(y, x)$ the value of the Hamiltonian H at the point C, $E = E(p, q, y, x) = H - h_C$, $S_i = S_i(y, x)$ are the areas of the regions G_i, $i = 1, 2$, $S_3 = S_1 + S_2$, $\Theta_i(y, x) = \{S_i, h_C\}$. Here and henceforth $\{\ ,\ \}$ are the Poisson brackets of the functions y, x: $\{f, g\} = f_x' g_y' - f_y' g_x'$.

The quantity $\varepsilon\Theta_i(y, x)$ is the rate of change of the area $S_i(y, x)$ in the adiabatic approximation, in the limit when the phase point in the region G_i approaches the separatrix. Below we assume that $\Theta_i(y, x) > \mathrm{const} > 0$ in the domain of variation of y, x. In this case we find that in the adiabatic approximation the phase points of G_3 may reach the separatrix in a finite slow time εt, and in the regions G_1 and G_2 they can leave the separatrix (since the "action" $I(H, y, x)$ is the area divided by 2π, bounded by the phase trajectory, and $I = \mathrm{const}$ in the adiabatic approximation).

The change in I, y, x can be described in an approximate manner using the following scheme /7/. Let the motion begin at $t = 0$, from the point $M_0(p_0, q_0, y_0, x_0)$, and $(p_0, q_0) \in G_3(y_0, x_0)$. In the region G_3 the AI is assumed constant until it reaches the separatrix: $I(H, y, x) = I^- = \mathrm{const}$. The variation in y, x is described by the solution $Y_3(\tau)$, $X_3(\tau)$, $\tau = \varepsilon t$ of the slow system with the Hamiltonian $\varepsilon\Phi(I^-, y, x)$ and initial conditions (y_0, x_0). The instant t_* of reaching the separatrix is found from the relation $S_3(Y_3(\tau_*), X_3(\tau_*)) = 2\pi I^-$, $\tau_* = \varepsilon t_*$. The quantities $y_* = Y_3(\tau_*)$, $x_* = X_3(\tau_*)$ can be determined by solving the system of equations $S_3(y_*, x_*) = 2\pi I^-$, $h_C(y_*, x_*) = H(p_0, q_0, y_0, x_0)$

After crossing the separatrix the point can continue its motion either in the region G_1, or in G_2. During its motion in G_i, $i = 1, 2$ the adiabatic invariant is again assumed constant $I = S_i(y_*, x_*)/(2\pi)$. The change of the slow variables is described by the solution $Y_i(\tau)$, $X_i(\tau)$ constructed for the region G_i of the slow system with initial condition $Y_i(\tau_*) = y_*$, $X_i(\tau_*) = x_*$.

When ε is small, the initial conditions in G_3 corresponding to the captures in G_1 and G_2 are very scrambled, therefore a capture by one or another region must be regarded as a random phenomenon. For the point M_0 the probability P_i of capture by the region G_i, $i = 1, 2$ is defined as the fraction of the phase volume of the smooth neighbourhood of the point M_0 trapped in G_i in the limit as $\varepsilon \to 0$, and the size of the neighbourhood $\delta \to 0$, $\varepsilon \ll \delta$ /5/ (first taken with respect to ε, and then with respect to δ). The probability is given by the formula

$$P_i = \Theta_i(y_*, x_*)/\Theta_3(y_*, x_*), \quad i = 1, 2$$

It was shown earlier* (*Neishtadt A.I. On certain resonance problems in non-linear systems. Candidate Dissertation, Moscow State University, 1976.) that the relations written for I, εt_*, y, x hold for the majority of initial conditions with an accuracy of $O(\varepsilon \ln \varepsilon)$ either for $i = 1$, or for $i = 2$. The sole set of initial conditions for which the above estimates do not hold, has a measure $O(\varepsilon^n)$, where $n \geqslant 1$ is any number specified in advance. (In analytic systems the sole set has a measure of $O(\exp(-c/\varepsilon))$, $c = \mathrm{const} > 0$.)

4. Asymptotic expansions for the rapid motion near the separatrices. The asymptotic expansions given above are analogous to the corresponding expansions from /2/, where a, b_i, d_i are smooth functions of y, x. We shall assume, to be specific, that $E > 0$ in the region G_3, $E < 0$ in the regions G_1 and G_2.

A. The following relations hold for the trajectory $E = h$, lying within the region G_i, $i = 1, 2, 3$:

$$\begin{aligned}
&T = -a_i \ln|h| + b_i + O(h \ln|h|),\ a_1 = a_2 = a,\ a_3 = 2a, \\
&b_3 = b_1 + b_2 \\
&2\pi I = S_i - a_i h \ln|h| + (b_i + a_i)h + O(h^2 \ln|h|) \\
&\oint\limits_{E=h} \frac{\partial E}{\partial \alpha}\,dt = -\frac{\partial S_i}{\partial \alpha} + O(h \ln|h|), \quad \alpha = y, x \\
&\oint\limits_{E=h} \left(\frac{dE}{dt}\right)_{\mathrm{tot}} dt = -\varepsilon\Theta_i + \varepsilon O(h \ln|h|)
\end{aligned} \tag{4.1}$$

In the last of the above formulas a derivative of the function E appears in the integrand by virtue of the complete initial system.

B. In order to obtain the asymptotic expansions it is convenient to rewrite the function u in (2.1) in the form

$$u = \frac{1}{4\pi}\left[\int\limits_0^{T}\left(\frac{\partial E}{\partial y}\int\limits_0^t \frac{\partial E}{\partial x}\,d\sigma\right)dt - \int\limits_0^T\left(\frac{\partial E}{\partial x}\int\limits_0^t \frac{\partial E}{\partial y}\,d\sigma\right)dt\right] + \frac{1}{2\pi}\int\limits_0^T\left(\frac{T}{2} - t\right)\left(\frac{\partial h_C}{\partial y}\frac{\partial E}{\partial x} - \frac{\partial h_C}{\partial x}\frac{\partial E}{\partial y}\right)dt$$

Let $C\eta\zeta$ be the system of principal coordinates for the saddle point C (Fig.1). If the point (p, q) lies in the region G_i, $i = 1, 2$ near C on the $C\eta$ axis, then the following expansion will hold for the function u:

$$2\pi u = d_i + O(\sqrt{|h|}\ln|h|),\ h = E(p, q, y, x)$$

If the point (p, q) lies in the region G_3, near C on the positive part of the $C\zeta$ axis, then

$$2\pi u = {}^1/_2 a(\Theta_2 - \Theta_1)\ln h + {}^1/_2(\Theta_1 b_2 - \Theta_2 b_1) + {}^1/_2\{S_2, S_1\} + d_3 + O(\sqrt{h}\ln h),\ d_3 = d_1 + d_2 \tag{4.2}$$

5. Computing the change in the AI. Away from the separatrices, the improved AI changes only by an amount of the order of $O(\varepsilon^2)$. Therefore, a change of the order of ε or less accumulates in the small neighbourhood of the separatrices, and the asymptotic expansions used in Sect.4 can by employed to calculate it. The arguments used follow, basically, those of /2/.

Let the phase point begin to move at $t = 0$ in the region G_3, and let this point lie, when $\tau = \tau^+$, in the region G_i, $i = 1$ or $i = 2$, with $I = I^+$, $J = J^+$. We denote by τ_*, y_*, x_* the values calculated in the adiabatic approximation of Sect.3, of the slow time and slow variables on reaching the separatrix (we assume that $\tau_* < \tau_+$); t_*^- is the instant of time at which the phase point arrives at the positive ray of the $C\zeta$ axis for the last time near the saddle point C; t_*^+ at the instant of time at which the phase point arrives at the $C\eta$ axis for the first time near the saddle point C; $h_*^\pm$, $J_*^\pm$, $y_*^\pm$, $x_*^\pm$ are the values of E, J, y, x on the trajectory when $t = t_*^\pm$; $\xi = h_*^-/(\varepsilon\Theta_3)$, $\xi_i = |h_*^+|/(\varepsilon\Theta_i)$.

Here Θ_j (and henceforth a, b_j, d_j, $\partial S_j/\partial x$, $\partial S_j/\partial y$, $j = 1, 2, 3$) are determined at $x = x_*$, $y = y_*$. We assume that the initial point does not belong to the exlusive set with a small measure for which the estimates of Sect.3 do not hold. Therefore $\varepsilon t_*^\pm = \tau_* + O(\varepsilon\ln\varepsilon)$.

The aim of subsequent discussion is to express, in the principal approximation, J^+ in terms of J^- and ξ_i.

5.1. Approaching the separatrix. When $0 \leqslant t \leqslant t_*^-$, the projection of the phase point on the p, q plane describes loops close to the unperturbed trajectories. It can be confirmed that during the motion in the region $E > h > \varepsilon$ the quantity J changes by an amount of the order of $O(\varepsilon^2/h)$. In particular in the region $E < \sqrt{\varepsilon}/|\ln\varepsilon|$ J varies by an amount of the order of $O(\varepsilon^{3/2}\ln\varepsilon)$. After reaching the region $0 < E < \sqrt{\varepsilon}/|\ln\varepsilon|$, we determine the instances of consecutive intersections of the ray $C\zeta$ near the point C by the moving point. We shall assign consecutive numbers to these instances, beginning with the last: $t_*^- = t_0 > t_1 > \ldots > t_N > 0$. We will denote the values of E, τ, I, J, y, x and $t = t_n$ by h_n, τ_n, I_n, J_n, y_n, x_n. If $\xi > k\sqrt{\varepsilon}$ where $k > 0$ is a sufficiently large constant, then the following formulas hold:

$$h_{n+1} = h_n + \varepsilon[\Theta_3 + O(\sqrt{h_{n+1}})],\quad \tau_{n+1} = \tau_n + \varepsilon[{}^1/_2 a\ln h_n + a\ln(h_n + \Theta_1\varepsilon) + {}^1/_2 a\ln h_{n+1} - b_3 + O(\sqrt{h_{n+1}})] \tag{5.1}$$

$$S_3(y_n, x_n) = S_3(y_0, x_0) + \Theta_3(\tau_n - \tau_0) + O(h_{n+1}^2\ln^2 h_{n+1}) + O(\varepsilon\sqrt{h_{n+1}}) \tag{5.2}$$

We write the change in the improved AI in the form $J_*^- - J^- = (J_*^- - J_N) + (J_N - J^-)$. The second term is of the order of $O(\varepsilon^{3/2}\ln\varepsilon)$. In computing the first term we use expansions (4.1) and (4.2) for J_N, J_*^-. The quantities h_N, $S_3(y_N, x_N)$ appearing in the expansion for J_N are obtained using (5.1). The manipulations are the same as those in /2/, and lead to the following result ($\Gamma(\cdot)$ is a gamma function):

$$2\pi(J_*^- - J^-) = 2\varepsilon a\Theta_3[-{}^1/_2\ln\{2\pi[\Gamma(\xi)\Gamma(\xi + \theta_{13})]^{-1}\} + \xi + (-\xi + {}^1/_2\theta_{23})\ln\xi] + O(\varepsilon^{3/2}\ln\varepsilon),\ \theta_{ij} = \Theta_i/\Theta_j \tag{5.3}$$

The estimation of the residual term is much better than in the intermediate formulas when $E = h_N$. This is due to the fact that the residual terms in asymptotic expansions are connected by relations ensuring adiabatic invariance for large h. The derivation of the residue term is time consuming and is omitted here, as in /2/.

Formula (5.3) enables us to obtain, in the principal approximation, a relation connecting $S_i(y_*^-, x_*^-)$, J^-, ξ $(i = 1, 2)$, which is needed in what follows. Indeed, the following relations hold:

$$S_j(y_*^-, x_*^-) - S_j(y_*, x_*) = \frac{\partial S_j}{\partial y}(y_*^- - y_*) + \frac{\partial S_j}{\partial x}(x_*^- - x_*) + O(\varepsilon^2\ln^2\varepsilon) \tag{5.4}$$

$$h_*^- = -\frac{\partial h_C}{\partial y}(y_*^- - y_*) - \frac{\partial h_C}{\partial x}(x_*^- - x_*) + O(\varepsilon^2\ln^2\varepsilon)$$

The last relation follows from the definition of y_*, x_* and the energy integral: $H = h_C(y_*, x_*)$. Regarding the first relation of (5.4) for $j = 3$ and the second relation as a linear system in $y_*^- - y_*$, $x_*^- - x_*$, solving it and substituting the result into the first relation for $j = 1, 2$, we can obtain

$$S_i(y_*^-, x_*^-) - S_i(y_*, x_*) = \{S_i, S_3\}h_*^-/\Theta_3 + \theta_{i3}(S_3(y_*^-, x_*^-) - S_3(y_*, x_*)) + O(\varepsilon^2 \ln^2 \varepsilon),\ i = 1, 2 \tag{5.5}$$

The expansions in Sect.4 make it possible to express $S_3(y_*^-, x_*^-)$ in terms of J_*^-, ξ; formula (5.3) gives J_*^- in terms of J^-, ξ, and therefore (5.5) enables us to express $S_i(y_*^-, x_*^-)$ in terms of J^-, ξ.

5.2. Passage across the separatrix. Estimates show that $\xi \in (0, 1 + k\sqrt{\varepsilon})$. If $\xi \in (k\sqrt{\varepsilon}, \theta_{23} - k\sqrt{\varepsilon})$, then after the passage across the separatrix the point becomes trapped in G_2, while if $\xi \in (\theta_{23} + k\sqrt{\varepsilon}, 1 - k\sqrt{\varepsilon})$ it is trapped in G_1. To be specific, we shall consider the first case. When $t_*^- \leqslant t \leqslant t_*^+$, the projection of the phase point on the p, q plane will describe a curve near the separatrix l_2. The following relations hold:

$$h_*^+ = h_*^- - \Theta_2\varepsilon + O(\varepsilon^{3/2})$$
$$t_*^+ = t_*^- - {}^1/_2 a \ln h_*^+ - {}^1/_2 a \ln |h_*^-| + b_2 + O(\varepsilon \ln^2 \varepsilon)$$
$$S_2(y_*^+, x_*^+) = S_2(y_*^-, x_*^-) + \Theta_2\varepsilon(t_*^+ - t_*^-) + O(\varepsilon^{3/2})$$

The relations together with the expansions of Sect.4, enable us to express J_*^+, ξ in terms of $S_2(y_*^-, x_*^-)$, ξ_2.

5.3. Moving away from the separatrix. Applying the arguments of Sect.5.1 to the motion in the region G_i, $i = 1, 2$, we obtain

$$2\pi(J^+ - J_*^+) = \varepsilon a \Theta_i[-\ln(\sqrt{2\pi}/\Gamma(\xi_i)) + \xi_i + ({}^1/_2 - \xi_i)\ln \xi_i] + O(\varepsilon^{3/2} \ln \varepsilon) \tag{5.6}$$

We note that formulas (5.3) and (5.6) hold for any number of degrees of freedom of the slow motion.

5.4. The final formula. In Sect.5.3 the quantity J^+ is expressed in terms of J_*^+, ξ_i. The formulas of Sect.5.2 enable us to express J_*^+, ξ in terms of $S_i(y_*^-, x_*^-)$, ξ_i. The formulas in Sect.5.1 enable us to express $S_i(y_*, x_*^-)$ in terms of J^-, ξ. As a result we can express J^+ in terms of J^-, ξ_i. When the region G_i is reached, we have, provided that $k\sqrt{\varepsilon} < \xi_i < 1 - k\sqrt{\varepsilon}$,

$$\begin{aligned}2\pi J^+ = {} & S_i(y_*, x_*) + \theta_{i3}(2\pi J^- - S_3(y_*, x_*)) + \\ & \varepsilon a\Theta_i(\xi_i - {}^1/_2)(\ln(\varepsilon\Theta_i) - 2\theta_{i3}\ln(\varepsilon\Theta_3)) - \\ & \varepsilon a\Theta_i \ln\{(2\pi)^{1/2}[\Gamma(\xi_i)\Gamma(\theta_{i3}(1 - \xi_i))\Gamma(1 - \theta_{i3}\xi_i)]^{-1}\} + \\ & \varepsilon\Theta_i({}^1/_2 - \xi_i)(b_i - \theta_{i3}b_3) + \varepsilon(d_i - \theta_{i3}d_3) + \\ & \varepsilon\theta_{i3}({}^1/_2 - \xi_i)\{S_i, S_3\} + O(\varepsilon^{3/2}(|\ln \varepsilon| + (1 - \xi_i)^{-1}))\end{aligned} \tag{5.7}$$

The quantity $\xi_i \in (0, 1)$ is a function of the initial conditions, whose value can be changed by an amount of the order of unity, by changing the arguments by a small amount of the order of ε. Therefore, it is best to treat ξ_i as a random quantity. For a given initial point $M_0(p_0, q_0, y_0, x_0)$ the probability that $\xi_i \in (\alpha, \beta) \subset (0, 1)$ is, by definition,

$$P_i^{(\alpha,\beta)} = \lim_{\delta\to 0} \lim_{\varepsilon\to 0} \operatorname{mes} U_{\delta,i}^{(\alpha,\beta)}/\operatorname{mes} U_{\delta,i}$$

where $U_{\delta,i}$ is the set of points in the δ-neighbourhood of M_0 entrapped within the region G_i, $U_{\delta,i}^{(\alpha,\beta)}$ is a set of points from $U_{\delta,i}$ for which $\xi_i \in (\alpha, \beta)$, mes $(\cdot)$ is the phase volume. It can be shown that $P_i^{(\alpha,\beta)} = \beta - \alpha$, i.e. the distribution ξ_i is uniform on $(0, 1)$. The quantity J^+ is also treated as random, and formula (5.7) determines its conditional distribution under the condition that the point is entrapped in the region G_i.

If the initial and final point of the trajectory are chosen so that $u = 0$ at these points, we can replace $J^\pm$ in (5.7) by $I^\pm$, while the second term on the right-hand side of (5.7) vanishes.

5.5. Formulas for describing the change in the AI during other passages. We have assumed above that $\theta_1 > 0$, $\theta_2 > 0$. When θ have different signs, we have other, different passages between the regions. In particular, let $\theta_1 > 0$, $\theta_2 < 0$, $\theta_3 > 0$. Then the points from the regions G_2 and G_3 arrive at the region G_1 with a probability of 1. The change in J during the passage from G_3 to G_1 is given by formula (5.7), and $k\sqrt{\varepsilon} < \xi_1 < \theta_{31} - k\sqrt{\varepsilon}$. The change in J during the passage from G_2 to G_1 is given, for $k\sqrt{\varepsilon} < \xi_2 < 1 - k\sqrt{\varepsilon}$, by the formula

$$2\pi J^+ = S_1(y_*, x_*) + \theta_{12}(2\pi J^- - S_2(y_*, x_*)) + \varepsilon a(1-\xi_2)\cdot$$
$$(\Theta_2 \ln(\varepsilon\Theta_1) - \Theta_1 \ln|\varepsilon\Theta_2|) - \varepsilon a\Theta_1 \ln\{2\pi(1-\xi_2)\cdot$$
$$\sqrt{\theta_{21}}\,[\Gamma(\xi_2)\,\Gamma(\theta_{31} - \theta_{21}\xi_2)]^{-1}\} + \varepsilon(1-\xi_2)(\Theta_1 b_2 - \Theta_2 b_1) +$$
$$\varepsilon(d_1 - \theta_{12}d_2) - \varepsilon(1-\xi_2)\{S_1, S_2\} + O(\varepsilon^{3/2}(|\ln\varepsilon| + (1-\xi_2)^{-1})),$$
$$\xi_2 = h_*^-/(\varepsilon\Theta_2) \tag{5.8}$$

Here h_*^- is the value of the function E during the last arrival at the $C\eta$ axis in the region G_2 near the saddle point C.

The formulas for the remaining versions of the passage are obtained from (5.7) and (5.8) by changing the directions, the time and numbering of the regions. The method of deriving formula (5.7) used here is quite general, and can be used to find the change in the AI for other types of phase patterns divided into regions by the separatrices. The essential assumptions here are that the saddle singularities are non-degenerate and that the rates of change of the regions in the adiabatic approximation are non-zero.

6. Example. The Hamiltonian of the bounded, plane elliptic three-body problem (the Sun, Jupiter, and an asteroid) near the 3:1 resonance, averaged over the longitudes of Jupiter and the asteroid, taking the above resonance into account, was reduced in /8/, in the principal approximation, to the form

$$H = {}^1/_2\alpha p^2 - A(x, y)\cos(q - Q(x, y)) - B(x, y) \tag{6.1}$$

Here p, y, q, $\varepsilon^{-1}x$ are the canonical variables, q is the mean longitude of the asteroid minus and triple mean longitude of Jupiter, x and $-y$ are proportional to the components of the Laplace vector for the asteroid, $\varepsilon = \sqrt{\mu}$, μ is the ratio of the masses of Jupiter and the Sun, $\alpha = \text{const} > 0$, the functions A, B are even in y and Q is odd in y, and $A \geqslant 0$.

The rapid system for (6.1) is a pendulum and the separatrices divide its phase pattern into the regions of forward rotation G_1, reverse rotation G_2, and oscillations G_3. The phase pattern of the slow system at the energy levels $H = r$ has, according to /8/, for some range of values of r, the form shown in Fig.2. The thick line shows the curve $L = \{x, y\colon h_C(y, x) = r\}$, $h_C = A - B$ corresponding to the separatrices and called in /8/ the indeterminacy curve. The curve separates the regions of the x, y, plane onto which the parts of the energy level corresponding to the regions $G_{1,2}$ and G_3 project naturally; here G_1 and G_2 project on the same finite region. The trajectories of the slow system outside L are represented by the lines $I(h, y, x) = \text{const}$. The function I becomes discontinuous on L, while the trajectories remain continuous. On reaching L the transitions from G_3 into G_1 and G_2 are equally probable. A saddle singularity exists in the phase pattern of the slow system. One of the separatrices passing through it intersects L. The region Σ in the phase pattern is filled with slow trajectories intersecting L.

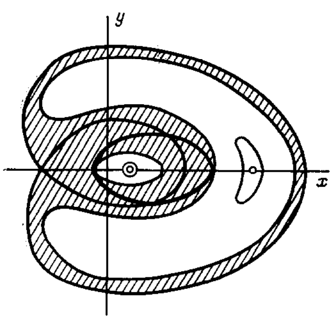

Fig.2

Applying the procedure of Sect.5 to the Hamiltonian (6.1), we see that the change in the improved AI during the passage from G_3 to G_i $(i = 1, 2)$ is given by the formula (5.7), in which we must put $S_1 = S_2 = {}^1/_2 S(y, x)$, $\Theta_1 = \Theta_2 = {}^1/_2\{S, h_C\}$, $b_1 = b_2$, $d_1 = -d_2 = d$ ($S = S(y, x)$ is the area of the region of oscillations for the pendulum).

Calculations lead to the formula

$$2\pi J_+ = \pi J_- - \varepsilon a\Theta \ln(2\sin(\pi\xi_i)) - (-1)^i\varepsilon d +$$
$$O(\varepsilon^{3/2}(|\ln\varepsilon| + (1-\xi_i)^{-1})),\quad i = 1, 2 \tag{6.2}$$
$$a = \frac{1}{\sqrt{\alpha A}},\quad \Theta = \frac{4}{\sqrt{\alpha A}}\{B, A\},\quad d = \frac{2\pi}{\alpha}\{B, Q\}$$

The coefficients a, Θ, d are calculated at the point (x_*, y_*) at which the trajectory of the slow system reaches the curve L, ξ_i is a quantity introduced in 5.4 and regarded as a random quantity uniformly distributed over the segment $(0, 1)$. In passing from (5.7) to (6.2), we used the complementing and doubling formulas for the gamma function.

The change in J during the passage from G_i to G_3 is given by formula (6.2), in which we must interchange the plus and minus signs and replace Θ by $-\Theta$. Therefore, the phase point, having emerged from G_3, passed through G_i and returned to G_3, obtains the following increment in the improved AI:

$$\Delta J = \frac{\varepsilon a\Theta}{\pi}\ln\frac{\sin\pi\xi_i'}{\sin\pi\xi_i} + O(\varepsilon^{3/2}(|\ln\varepsilon| + (1-\xi_i)^{-1} + (1-\xi_i')^{-1})) \tag{6.3}$$

The coefficients a, Θ are calculated at the point (x_*, y_*) of emergence on the curve of indeterminacy using the passage from G_3 to G_i; ξ_i, and ξ_i' are the quantities ξ_i introduced in Sect.5.4 for the passages from G_3 to G_i and from G_i to G_3 respectively. We take into account the fact that these two passages take place at the points of the x, y plane symmetrical with respect to the axis $y = 0$, and the values of $a\Theta$ at these points have different signs, while the values of d are the same. In the limit, as $\varepsilon \to 0$, the quantities ξ_i and ξ_i' are regarded as random and uniformly distributed over the segment $(0, 1)$, and it can be shown that they are independent. Also, as $\varepsilon \to 0$, the quantity $\varepsilon^{-1}\Delta J$ is regarded as random and formula (6.3) yields its distribution law. According to (6.3) the quantity has zero mean and variance

$$\sigma^2 = 2a^2\Theta^2\pi^{-2}\int_0^1 \ln^2(2\sin\pi\xi)\,d\xi \approx 0{,}17a^2\Theta^2$$

In the case of multiple passages across the separatrix the summation of the quasirandom changes in the AI results in diffusion, discovered in /8/ by numerical integration. Although in the course of diffusion the projection of the phase point onto the plane of slow variables intersects the separatrix of slow motion, the nature of the slow motion changes sharply. In particular, for the real parameters of the sun-Jupiter system the eccentricity of the asteroid can increase from values less than 0.1 to about 0.4 for which the asteroid will intersect the orbit of Mars. The perturbing influence of Mars ejects such asteroids from the main belt. According to /8/ a gap forms in the distribution of the asteroids, close to the observed Kirkwood gap at the 3:1 resonance.

There is no rigorous theory of the diffusion of the AI. It is probable that the small neighbourhood of the initial point spreads out due to diffusion over the region Σ over the $N \sim \varepsilon^{-2}\sigma_L^{-2}I_L^2$ passage across the separatrix, and this requires a time of $t_L \sim \varepsilon^{-3}\sigma_L^{-2}I_L^2\tau_L$.
Here I_L is the change in I along L, and σ_L, τ_L are the characteristic values of the variance σ^2 and of the slow time interval between the passages. The quantity t_L in theory /8/ is identical with the characteristic time of formation of the gap in the distribution of the asteroids. In the case of real parameters of the Sun-Jupiter system, we find that $t_L \sim 10^5 - 10^6$ years, and this agrees, in order of magnitude, with the results of numerical integration given in /8/.

REFERENCES

1. ARNOL'D V.I., Mathematical Methods of Classical Mechanics. Moscow, Nauka, 1974.
2. NEISHTADT A.I., On the change of the adiabatic invariant on crossing a separatrix. Fizika plazmy, 12, 8, 1986.
3. TENNYSON J.L., CARY J.R. and ESCANDE D.F., Change of the adiabatic invariant due to separatrix crossing, Phys. Rev. Lett, 56, 20, 1986.
4. TIMOFEYEV A.V., On the question of the constancy of the adiabatic invariant during a change in the character of the motion. Zh. Eksper. Teor. Fiz., 75, 4, 1978.
5. ARNOL'D V.I., Small denominators and problems of stability in classical and celestial mechanics. Uspekhi Mat. Nauk, 18, 6, 1963.
6. BOGOLYUBOV N.N. and MITROPOL'SKII YU.A., Asymptotic Methods in the Theory of Non-linear Oscillations. Moscow, Nauka, 1974.
7. LIFSHITZ I.M., SLUTSKIN A.A. and NABUTOVSKII V.M., On the phenomenon of "diffusion" of charged quasiparticles on singularities in p-space. Dokl. Akad. Nauk SSSR, 137, 3, 1961.
8. WISDOM J., A perturbative treatment of motion near the 3/1 commensurability, Icarus, 63, 2, 1985.

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